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6 THE IMAGINARY THAT ISN'T

When I was a mere slip of a lad and attended college, I had a friend with whom I ate lunch every day. His 11 A.M. class was in sociology, which I absolutely refused to take, and my 11 A.M. class was calculus, which he as steadfastly refused to take—so we had to separate at eleven and meet at twelve.

As it happened, his sociology professor was a scholar who did things in the grand manner, holding court after class was over. The more eager students gathered close and listened to him pontificate for an additional fifteen minutes, while they threw in an occasional log in the form of a question to feed the flame of oracle.

Consequently, when my calculus lecture was over, I had to enter the sociology room and wait patiently for court to conclude.

Once I walked in when the professor was listing on the board his classification of mankind into the two groups of mystics and realists, and under mystics he had included the mathematicians along with the poets and theologians. One student wanted to know why.

"Mathematicians," said the professor, "are mystics because they believe in numbers that have no reality."

Now ordinarily, as a nonmember of the class, I sat in the corner and suffered in silent boredom, but now I rose convulsively, and said, "What numbers?"

The professor looked in my direction and said, "The square root of minus one. It has no existence. Mathematicians call it imaginary. But they believe it has some kind of existence in a mystical way."

"There's nothing mystical about it," I said, angrily. "The square root of minus one is just as real as any other number."

The professor smiled, feeling he had a live one on whom he could now proceed to display his superiority of intellect (I have since had classes of my own and I know exactly how he felt). He said, silkily, "We have a young mathematician here who wants to prove the reality

of the square root of minus one. Come, young man, hand me the square root of minus one pieces of chalk!"

I reddened, "Well, now, wait—"

"That's all," he said, waving his hand. Mission, he imagined, accomplished, both neatly and sweetly.

But I raised my voice. "I'll do it. I'll do it. I'll hand you the square root of minus one pieces of chalk, if you hand me a one-half piece of chalk."

The professor smiled again, and said, "Very well," broke a fresh piece of chalk in half, and handed me one of the halves. "Now for your end of the bargain."

"Ah, but wait," I said, "you haven't fulfilled your end. This is one piece of chalk you've handed me, not a one-half piece." I held it up for the others to see. "Wouldn't you all say this was one piece of chalk? It certainly isn't two or three."

Now the professor wasn't smiling. "Hold it. One piece of chalk is a piece of regulation length. You have one that's half the regulation length."

I said, "Now you're springing an arbitrary definition on me. But even if I accept it, are you willing to maintain that this is a one-half piece of chalk and not a 0.48 piece or a 0.52 piece? And can you really consider yourself qualified to discuss the square root of minus one, when you're a little hazy on the meaning of one half?"

But by now the professor had lost his equanimity altogether and his final argument was unanswerable. He said, "Get the hell out of here!" I left (laughing) and thereafter waited for my friend in the corridor.

Twenty years have passed since then and I suppose I ought to finish the argument—

Let's start with a simple algebraic equation such as $x + 3 = 5$. The expression, x, represents some number which, when substituted for x, makes the expression a true equality. In this particular case x must equal 2, since $2 + 3 = 5$, and so we have "solved for x."

The interesting thing about this solution is that it is the *only* solution. There is no number but 2 which will give 5 when 3 is added to it.

This is true of any equation of this sort, which is called a "linear equation" (because in geometry it can be represented as a straight line) or "a polynomial equation of the first degree." No polynomial equation of the first degree can ever have more than one solution for x.

There are other equations, however, which *can* have more than one solution. Here's an example: $x^2 - 5x + 6 = 0$, where x^2 ("x square" or "x squared") represents x times x. This is called a "quadratic equation," from a Latin word for "square," because it involves x square. It is also called "a polynomial equation of the second degree" because of the little ^{2} in x^2. As for x itself, that could be written x^1, except that the ^{1} is always omitted and taken for granted, and that is why $x + 3 = 5$ is an equation of the first degree.

If we take the equation $x^2 - 5x + 6 = 0$, and substitute 2 for x, then x^2 is 4, while $5x$ is 10, so that the equation becomes $4 - 10 + 6 = 0$, which is correct, making 2 a solution of the equation.

However, if we substitute 3 for x, then x^2 is 9 and $5x$ is 15, so that the equation becomes $9 - 15 + 6 = 0$, which is also correct, making 3 a second solution of the equation.

Now no equation of the second degree has ever been found which has more than two solutions, but what about polynomial equations of the third degree? These are equations containing x^3 ("x cube" or "x cubed"), which are therefore also called "cubic equations." The expression x^3 represents x times x times x.

The equation $x^3 - 6x^2 + 11x - 6 = 0$ has three solutions, since you can substitute 1, 2, or 3 for x in this equation and come up with a true equality in each case. No cubic equation has ever been found with more than three solutions, however.

In the same way polynomial equations of the fourth degree can be constructed which have four solutions but no more; polynomial equations of the fifth degree, which have five solutions but no more; and so on. You might say, then, that a polynomial equation of the nth degree can have as many as n solutions, but never more than n.

Mathematicians craved something even prettier than that and by about 1800 found it. At that time, the German mathematician Karl Friedrich Gauss showed that every equation of the nth degree had exactly n solutions, not only no more, but also no less.

However, in order to make the fundamental theorem true, our no-

tion of what constitutes a solution to an algebraic equation must be drastically enlarged.

To begin with, men accept the "natural numbers" only: 1, 2, 3, and so on. This is adequate for counting objects that are only considered as units generally. You can have 2 children, 5 cows, or 8 pots; while to have $2\frac{1}{2}$ children, $5\frac{1}{4}$ cows, or $8\frac{1}{3}$ pots does not make much sense.

In measuring continuous quantities such as lengths or weights, however, fractions became essential. The Egyptians and Babylonians managed to work out methods of handling fractions, though these were not very efficient by our own standards; and no doubt conservative scholars among them sneered at the mystical mathematicians who believed in a number like $5\frac{1}{2}$, which was neither 5 nor 6.

Such fractions are really ratios of whole numbers. To say a plank of wood is $2\frac{5}{8}$ yards long, for instance, is to say that the length of the plank is to the length of a standard yardstick as 21 is to 8. The Greeks, however, discovered that there were definite quantities which could not be expressed as ratios of whole numbers. The first to be discovered was the square root of 2, commonly expressed as $\sqrt{2}$, which is that number which, when multiplied by itself, gives 2. There is such a number but it cannot be expressed as a ratio; hence, it is an "irrational number."

Only thus far did the notion of number extend before modern times. Thus, the Greeks accepted no number smaller than zero. How can there be less than nothing? To them, consequently, the equation $x + 5 = 3$ had no solution. How can you add 5 to any number and have 3 as a result? Even if you added 5 to the smallest number (that is, to zero), you would have 5 as the sum, and if you added 5 to any other number (which would have to be larger than zero), you would have a sum greater than 5.

The first mathematician to break this taboo and make systematic use of numbers less than zero was the Italian, Girolamo Cardano. After all, there *can* be less than nothing. A debt is less than nothing.

If all you own in the world is a two-dollar debt, you have two dollars less than nothing. If you are then given five dollars, you end with three dollars of your own (assuming you are an honorable man who pays his debts). Consequently, in the equation $x+5=3$, x can be set equal to -2, where the minus sign indicates a number less than zero.

Such numbers are called "negative numbers," from a Latin word meaning "to deny," so that the very name carries the traces of the Greek denial of the existence of such numbers. Numbers greater than zero are "positive numbers" and these can be written $+1$, $+2$, $+3$, and so on.

From a practical standpoint, extending the number system by including negative numbers simplifies all sorts of computations; as, for example, those in bookkeeping.

From a theoretical standpoint, the use of negative numbers means that every equation of the first degree has exactly one solution. No more; no less.

If we pass on to equations of the second degree, we find that the Greeks would agree with us that the equation $x^2-5x+6=0$ has two solutions, 2 and 3. They would say, however, that the equation $x^2+4x-5=0$ has only one solution, 1. Substitute 1 for x and x^2 is 1, while $4x$ is 4, so that the equation becomes $1+4-5=0$. No other number will serve as a solution, as long as you restrict yourself to positive numbers.

However, the number -5 is a solution, if we consider a few rules that are worked out in connection with the multiplication of negative numbers. In order to achieve consistent results, mathematicians have decided that the multiplication of a negative number by a positive number yields a negative product, while the multiplication of a negative number by a negative number yields a positive product.

If, in the equation $x^2+4x-5=0$, -5 is substituted for x, then x^2 becomes -5 times -5, or $+25$, while $4x$ becomes $+4$ times -5, or -20. The equation becomes $25-20-5=0$, which is true. We would say, then, that there are two solutions to this equation, $+1$ and -5.

Sometimes, a quadratic equation does indeed seem to have but a single root, as, for example, $x^2-6x+9=0$, which will be a true equality if and only if the number $+3$ is substituted for x. However, the mechanics of solution of the equation show that there are actually two solutions, which happen to be identical. Thus, $x^2-6x+9=0$ can

be converted to $(x-3)(x-3)=0$ and each $(x-3)$ yields a solution. The two solutions of this equation are, therefore, $+3$ and $+3$.

Allowing for occasional duplicate solutions, are we ready to say then that all second-degree equations can be shown to have exactly two solutions if negative numbers are included in the number system?

Alas, no! For what about the equation $x^2+1=0$. To begin with, x^2 must be -1, since substituting -1 for x^2 makes the equation $-1+1=0$, which is correct enough.

But if x^2 is -1, then x must be the famous square root of minus one ($\sqrt{-1}$), which occasioned the set-to between the sociology professor and myself. The square root of minus one is that number which when multiplied by itself will give -1. But there is no such number in the set of positive and negative quantities, and that is the reason the sociology professor scorned it. First, $+1$ times $+1$ is $+1$; secondly, -1 times -1 is $+1$.

To allow any solution at all for the equation $x^2+1=0$, let alone two solutions, it is necessary to get past this roadblock. If no positive number will do and no negative one either, it is absolutely essential to define a completely new kind of number; an imaginary number, if you like; one with its square equal to -1.

We could, if we wished, give the new kind of number a special sign. The plus sign does for positives and the minus sign for negatives; so we could use an asterisk for the new number and say that *1 ("star one") times *1 was equal to -1.

However, this was not done. Instead, the symbol i (for "imaginary") was introduced by the Swiss mathematician Leonhard Euler in 1777 and was thereafter generally adopted. So we can write $i=\sqrt{-1}$ or $i^2=-1$.

Having defined i in this fashion, we can express the square root of any negative number. For instance, $\sqrt{-4}$ can be written $\sqrt{4}$ times $\sqrt{-1}$, or $2i$. In general, any square root of a negative number, $\sqrt{-n}$, can be written as the square root of the equivalent positive number times the square root of minus one; that is, $\sqrt{-n}=\sqrt{n}\,i$.

In this way, we can picture a whole series of imaginary numbers exactly analogous to the series of ordinary or "real numbers." For 1, 2, 3, 4, . . . , we would have i, $2i$, $3i$, $4i$. . . . This would include fractions, for $\frac{2}{3}$ would be matched by $\frac{2i}{3}$; $\frac{15}{17}$ by $\frac{15i}{17}$, and so on. It would also include irrationals, for $\sqrt{2}$ would be matched by $\sqrt{2}\,i$ and even a number like π (pi) would be matched by πi.

These are all comparisons of positive numbers with imaginary numbers. What about negative numbers? Well, why not negative imaginaries, too? For $-1, -2, -3, -4, \ldots$, there would be $-i, -2i, -3i, -4i, \ldots$

So now we have four classes of numbers: 1) positive real numbers, 2) negative real numbers, 3) positive imaginary numbers, 4) negative imaginary numbers. (When a negative imaginary is multiplied by a negative imaginary, the product is negative.)

Using this further extension of the number system, we can find the necessary two solutions for the equation $x^2 + 1 = 0$. They are $+i$ and $-i$. First $+i$ times $+i$ equals -1, and secondly $-i$ times $-i$ equals -1, so that in either case, the equation becomes $-1 + 1 = 0$, which is a true equality.

In fact, you can use the same extension of the number system to find all four solutions for an equation such as $x^4 - 1 = 0$. The solutions are $+1$, -1, $+i$, and $-i$. To show this, we must remember that any number raised to the fourth power is equal to the square of that number multiplied by itself. That is, n^4 equals n^2 times n^2.

Now let's substitute each of the suggested solutions into the equations so that x^4 becomes successively $(+1)^4$, $(-1)^4$, $(+i)^4$, and $(-i)^4$.

First $(+1)^4$ equals $(+1)^2$ times $(+1)^2$, and since $(+1)^2$ equals $+1$, that becomes $+1$ times $+1$, which is $+1$.

Second, $(-1)^4$ equals $(-1)^2$ times $(-1)^2$, and since $(-1)^2$ also equals $+1$, the expression is again $+1$ times $+1$, or $+1$.

Third, $(+i)^4$ equals $(+i)^2$ times $(+i)^2$ and we have defined $(+i)^2$ as -1, so that the expression becomes -1 times -1, or $+1$.

Fourth, $(-i)^4$ equals $(-i)^2$ times $(-i)^2$, which is also -1 times -1, or $+1$.

All four suggested solutions, when substituted into the equation $x^4 - 1 = 0$, give the expression $+1 - 1 = 0$, which is correct.

It might seem all very well to talk about imaginary numbers—for a mathematician. As long as some defined quantity can be made subject to rules of manipulation that do not contradict anything else in the mathematical system, the mathematician is happy. He doesn't really care what it "means."

Ordinary people do, though, and that's where my sociologist's charge of mysticism against mathematicians arises.

And yet it is the easiest thing in the world to supply the so-called "imaginary" numbers with a perfectly real and concrete significance. Just imagine a horizontal line crossed by a vertical line and call the point of intersection zero. Now you have four lines radiating out at mutual right angles from that zero point. You can equate those lines with the four kinds of numbers.

If the line radiating out to the right is marked off at equal intervals, the marks can be numbered $+1, +2, +3, +4, \ldots$, and so on for as long as we wish, if we only make the line long enough. Between the markings are all the fractions and irrational numbers. In fact, it can be shown that to every point on such a line there corresponds one and only one positive real number, and for every positive real number there is one and only one point on the line.

The line radiating out to the left can be similarly marked off with the negative real numbers, so that the horizontal line can be considered the "real-number axis," including both positives and negatives.

Similarly, the line radiating upward can be marked off with the positive imaginary numbers, and the one radiating downward with the negative imaginary numbers. The vertical line is then the imaginary-number axis.

Suppose we label the different numbers not by the usual signs and symbols, but by the directions in which the lines point. The rightward line of positive real numbers can be called East because that would be its direction of extension on a conventional map. The leftward line of negative real numbers would be West; the upward line of positive imaginaries would be North; and the downward line of negative imaginaries would be South.

Now if we agree that $+1$ times $+1$ equals $+1$, and if we concentrate on the compass signs as I have defined them, we are saying that East times East equals East. Again since -1 times -1 also equals $+1$, West times West equals East. Then, since $+i$ times $+i$ equals -1, and so does $-i$ times $-i$, then North times North equals West and so does South times South.

We can also make other combinations such as -1 times $+i$, which equals $-i$ (since positive times negative yields a negative product even when imaginaries are involved), so that West times North equals South. If we list all the possible combinations as compass points, abbreviating those points by initial letters, we can set up the following system:

E × E = E	E × S = S	E × W = W	E × N = N
S × E = S	S × S = W	S × W = N	S × N = E
W × E = W	W × S = N	W × W = E	W × N = S
N × E = N	N × S = E	N × W = S	N × N = W

There is a very orderly pattern here. Any compass point multiplied by East is left unchanged, so that East as a multiplier represents a rotation of 0°. On the other hand, any compass point multiplied by West is rotated through 180° ("about face"). North and South represent right-angle turns. Multiplication by South results in a 90° clockwise turn ("right face"); while multiplication by North results in a 90° counterclockwise turn ("left face").

Now it so happens that an unchanging direction is the simplest arrangement, so East (the positive real numbers) is easier to handle and more comforting to the soul than any of the others. West (the negative real numbers), which produces an about face but leaves one on the same line at least, is less comforting, but not too bad. North and South (the imaginary numbers), which send you off in a new direction altogether, are least comfortable.

But viewed as compass points, you can see that no set of numbers is more "imaginary" or, for that matter, more "real" than any other.

Now consider how useful the existence of two number axes can be. As long as we deal with the real numbers only, we can move along the real-number axis, backward and forward, one-dimensionally. The same would be true if we used only the imaginary-number axis.

Using both, we can define a point as so far right or left on the real-number axis and so far up or down on the imaginary-number axis. This will place the point somewhere in one of the quadrants formed by the two axes. This is precisely the manner in which points

are located on the earth's surface by means of latitude and longitude.

We can speak of a number such as $+5 + 5i$, which would represent the point reached when you marked off 5 units East followed by 5 units North. Or you can have $-7 + 6i$ or $+0.5432 - 9.115i$ or $+\sqrt{2} + \sqrt{3}\, i$.

Such numbers, combining real and imaginary units, are called "complex numbers."

Using both axes, any point in a plane (and not merely on a line) can be made to correspond to one and only one complex number. Again every conceivable complex number can be made to correspond to one and only one point on a plane.

In fact, the real numbers themselves are only special cases of the complex numbers, and so, for that matter, are the imaginary numbers. If you represent complex numbers as all numbers of the form $+a + bi$, then the real numbers are all those complex numbers in which b happens to be equal to zero. And imaginary numbers are all the complex numbers in which a happens to be equal to zero.

The use of the plane of complex numbers, instead of the lines of real numbers only, has been of inestimable use to the mathematician.

For instance, the number of solutions in a polynomial equation is equal to its degree only if complex numbers are considered as solutions, rather than merely real numbers and imaginary numbers. For instance the two solutions of $x^2 - 1 = 0$ are $+1$ and -1, which can be written as $+1 + 0i$ and $-1 + 0i$. The two solutions of $x^2 + 1 = 0$ are $+i$ and $-i$, or $0 + i$ and $0 - i$. The four solutions of $x^4 - 1 = 0$ are all four complex numbers just listed.

In all these very simple cases, the complex numbers contain zeros and boil down to either real numbers or to imaginary numbers. This, nevertheless, is not always so. In the equation $x^3 - 1 = 0$ one solution, to be sure, is $+1 + 0i$ (which can be written simply as $+1$), but the other two solutions are $-\frac{1}{2} + \frac{1}{2}\sqrt{3}\, i$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{3}\, i$.

The Gentle Reader with ambition can take the cube of either of these expressions (if he remembers how to multiply polynomials algebraically) and satisfy himself that it will come out $+1$.

Complex numbers are of practical importance too. Many familiar measurements involve "scalar quantities" which differ only in magnitude. One volume is greater or less than another; one weight is greater or less than another; one density is greater or less than another. For that matter, one debt is greater or less than another. For all such measurements, the real numbers, either positive or negative, suffice.

However, there are also "vector quantities" which possess both magnitude and direction. A velocity may differ from another velocity not only in being greater or less, but in being in another direction. This holds true for forces, accelerations, and so on.

For such vector quantities, complex numbers are necessary to the mathematical treatment, since complex numbers include both magnitude and direction (which was my reason for making the analogy between the four types of numbers and the compass points).

Now, when my sociology professor demanded "the square root of minus one pieces of chalk," he was speaking of a scalar phenomenon for which the real numbers were sufficient.

On the other hand, had he asked me how to get from his room to a certain spot on the campus, he would probably have been angered if I had said, "Go two hundred yards." He would have asked, with asperity, "In which direction?"

Now, you see, he would have been dealing with a vector quantity for which the real numbers are insufficient. I could satisfy him by saying "Go two hundred yards northeast," which is equivalent to saying "Go $100\sqrt{2}$ plus $100\sqrt{2}\,i$ yards."

Surely it is as ridiculous to consider the square root of minus one "imaginary" because you can't use it to count pieces of chalk as to consider the number 200 as "imaginary" because by itself it cannot express the location of one point with reference to another.

7 PRE-FIXING IT UP

I go through life supported and bolstered by many comforting myths, as do all of us. One of my own particularly cherished articles of faith is that there are no arguments against the metric system and that the common units make up an indefensible farrago of nonsense that we keep out of stubborn folly.

Imagine the sobering effect, then, of having recently come across a letter by a British gentleman who bitterly denounced the metric system as being artificial, sterile, and not geared to human needs. For instance, he said (and I don't quote exactly), if one wants to drink beer, a pint of beer is the thing. A liter of beer is too much and half a liter is too little, but a pint, ah, that's just right.[1]

As far as I can tell, the gentleman was serious in his provincialism, and in considering that that to which he is accustomed has the force of a natural law. It reminds me of the pious woman who set her face firmly against all foreign languages by holding up her Bible and saying, "If the English language was good enough for the prophet Isaiah, and the apostle Paul, it is good enough for me."

But mainly it reminds me that I want to write an essay on the metric system.

In order to do so, I want to begin by explaining that the value of the system does not lie in the actual size of the basic units. Its worth is this: that it is a logical *system.* The units are sensibly interrelated.

All other sets of measurements with which I am acquainted use separate names for each unit involving a particular type of quantity. In distance, we ourselves have miles, feet, inches, rods, furlongs, and so on. In volume, we have pecks, bushels, pints, drams. In weight,

[1] Before you write to tell me that half a liter is larger than a pint, let me explain that though it is larger than an American pint, it is smaller than a British pint.